

Filters

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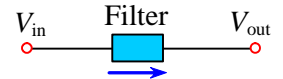
Introduction

- Signals often contain noise and/or frequency components that we want to *eliminate* or *suppress (attenuate)*.
- In this learning module, we discuss **filters** – what they do, how they work, and how to build them.
- Most of the filters discussed here are **passive filters** – they do not involve operational amplifiers and feedback loops, as do **active filters**, which are discussed briefly in a separate module.

Types of filters

- A **filter** is used to **remove some unwanted frequency components in a voltage signal**.

- Consider a filter that modifies an input voltage signal V_{in} to produce an output voltage V_{out} , as sketched to the right.



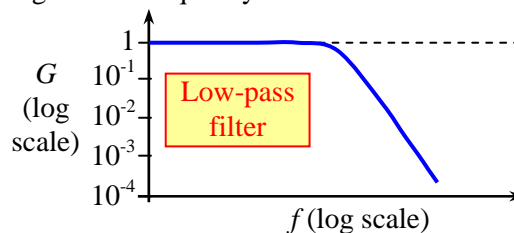
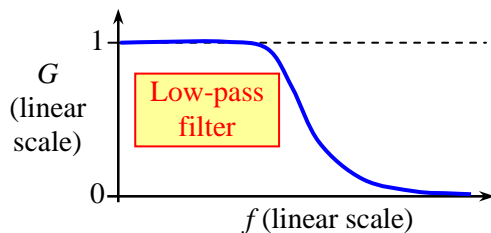
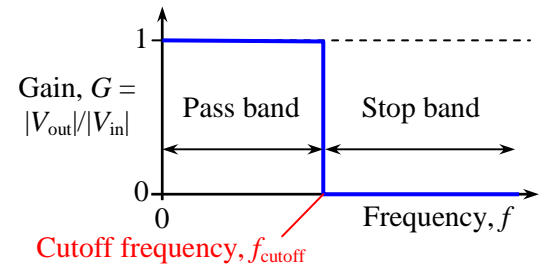
- We define the **gain G** of the filter as **the ratio of the magnitude of the output voltage to the magnitude of the input voltage**, $G = |V_{out}|/|V_{in}|$.

- The gain of a filter typically ranges from 0 to 1.

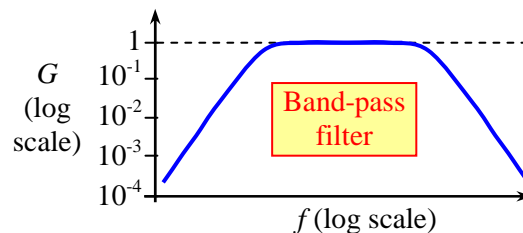
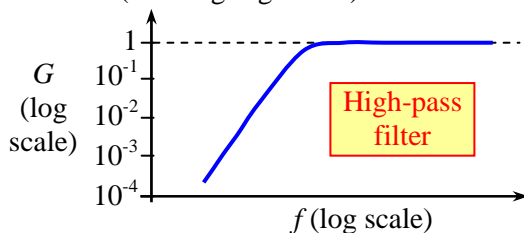
- We categorize a filter according to its **frequency response diagram** – a plot of gain G vs. frequency f .

- There are four basic types of filters:

- A **low-pass filter** lets low frequencies go through or *pass*, but attenuates or cuts off high frequencies. The frequency response diagram for an **ideal low-pass filter** is sketched to the right.
- The **range of frequencies that is passed by the filter** is called the **pass band**. Ideally, $V_{out}/V_{in} = 1$ in the pass band – the output voltage is not affected in any way by the filter.
- The **range of frequencies that is cut off or stopped by the filter** is called the **stop band**. Ideally, $V_{out}/V_{in} = 0$ in the stop band – the output voltage is completely cut off by the filter.
- Real low-pass filters are *not* ideal, and do not cut off the high frequencies abruptly. Instead, there is a **gradual roll off** of the gain from 1 to 0. A typical frequency response diagram for a low-pass filter is sketched below left with linear scales for both gain and frequency.

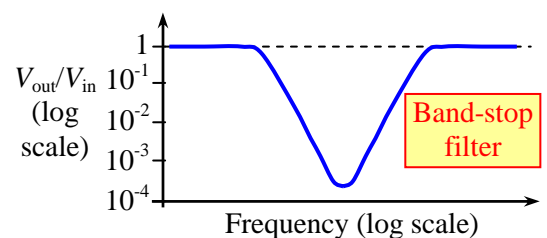


- Logarithmic scales are usually used when plotting the frequency response diagram – on both the horizontal and vertical axes, as sketched above right.
- A **high-pass filter** lets high frequencies go through or *pass*, but attenuates low frequencies, as sketched below left (with log-log scales).



- A **band-pass filter** is a combination of the above two. It lets a band of frequencies go through or *pass*, but attenuates both low frequencies *and* high frequencies, as sketched above right.

- Finally, a **band-stop filter** is the opposite of a band-pass filter. It lets *all* frequencies go through or *pass*, except for some band of frequencies, which it suppresses or *stops*, as sketched to the right.

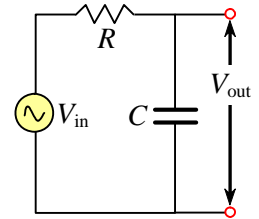


Passive filter circuits

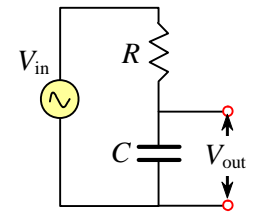
- In this section, some simple passive filter circuits are constructed and analyzed.
- Resistors, capacitors, and inductors are the only components used to construct passive filters. As mentioned in the introduction, *active filters* use operational amplifiers and feedback circuits, and are not discussed here.

First-order passive low-pass filter

- A resistor and capacitor can be used together to create a simple *first-order passive low-pass filter*, whose circuit is shown to the right.
- Consider an input voltage that is a pure sine wave of frequency f (or angular frequency $\omega = 2\pi f$), with no DC offset or phase shift – it is an AC voltage.
- The input voltage is of the form $V_{in} = V_p \sin(2\pi ft) = V_p \sin(\omega t)$, where V_p is the *peak voltage* or *amplitude* A of the signal, which is also equal to $|V_{in}|$. *Note:* The *peak-to-peak amplitude* V_{p-to-p} is equal to twice V_p , i.e., $V_{p-to-p} = 2V_p$.



- Output voltage V_{out} is measured by some device (voltmeter, oscilloscope, etc.).
- It is always assumed that **the device that measures output voltage V_{out} has infinite impedance**. This means that the measuring device does not affect the circuit in any way. For example, it does not draw any current or cause any voltage drop. In other words, it is a *non-intrusive* measuring device.
- Modern digital multimeters (DMMs), oscilloscopes, and PC data acquisition cards have huge, but not infinite impedance, so the above assumption is very good.
- To analyze the low-pass filter circuit, we think of it as a simple *voltage divider*, except with one of the resistors replaced by a capacitor, as sketched to the right.
- Instead of resistance, *impedance* is used to analyze this divider circuit.
- Recall that for the resistor, $Z_R = R$, and for the capacitor, $Z_C = 1/(i\omega C)$, where we use **bold** fonts to indicate complex variables.



- Impedance adds in series just like resistance, so $Z_{total} = Z_R + Z_C = R + \frac{1}{i\omega C}$.
- Just as for a simple resistor voltage divider circuit, the output voltage here is equal to the input voltage times a linear fraction of the impedances as follows (note that the **bolded** output voltage V_{out} is complex):

$$V_{out} = V_{in} \frac{Z_C}{Z_R + Z_C} = V_{in} \frac{1/i\omega C}{R + (1/i\omega C)}$$

- After rearranging and multiplying and dividing by the complex conjugate of the denominator, the complex voltage is split into its real and imaginary components, $V_{out} = V_{in} \frac{1}{1 + i\omega RC} = V_{in} \frac{1 - i\omega RC}{(1 + i\omega RC)(1 - i\omega RC)} =$

$$V_{in} \left[\frac{1}{1 + (\omega RC)^2} - i \frac{\omega RC}{1 + (\omega RC)^2} \right]$$

- It turns out that **the magnitude of V_{out} is the magnitude of the output voltage V_{out} itself**, and **the angle of V_{out} in the complex plane is the phase shift ϕ of V_{out} compared the input signal V_{in}** .
- Recall that the magnitude of a complex number is a real number called the *modulus*. Mathematically, we find the magnitude or modulus of a complex number by taking the square root of the sum of the real part squared and the imaginary part squared, $|V_{out}| = |V_{out}| = |V_{in}| \sqrt{\frac{1}{[1 + (\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1 + (\omega RC)^2]^2}} =$

$$|V_{in}| \sqrt{\frac{1 + (\omega RC)^2}{[1 + (\omega RC)^2]^2}} = |V_{in}| \sqrt{\frac{1}{1 + (\omega RC)^2}}, \text{ or finally, } |V_{out}| = |V_{in}| \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

- The *cutoff radian frequency* (also called the *corner radian frequency*, or sometimes the *break radian frequency*) for this simple passive RC low-pass filter circuit is defined as $\omega_{cutoff} = \frac{1}{RC}$, noting that this is the *radian frequency* (radians per second), not the *physical frequency* (Hz).

- Since $\omega = 2\pi f$, the **physical cutoff frequency** f_{cutoff} is defined as $f_{\text{cutoff}} = \frac{\omega_{\text{cutoff}}}{2\pi} = \frac{1}{2\pi RC}$.

- Then, the **magnitude of the output voltage** is re-written in terms of radian frequencies as

$$|V_{\text{out}}| = |V_{\text{in}}| \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_{\text{cutoff}}}\right)^2}}$$

$$\text{or in terms of physical frequencies as } |V_{\text{out}}| = |V_{\text{in}}| \frac{1}{\sqrt{1 + \left(\frac{f}{f_{\text{cutoff}}}\right)^2}}$$

- For this first-order passive low-pass filter, the gain is

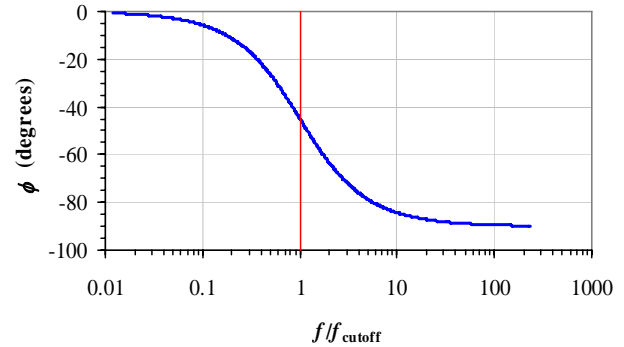
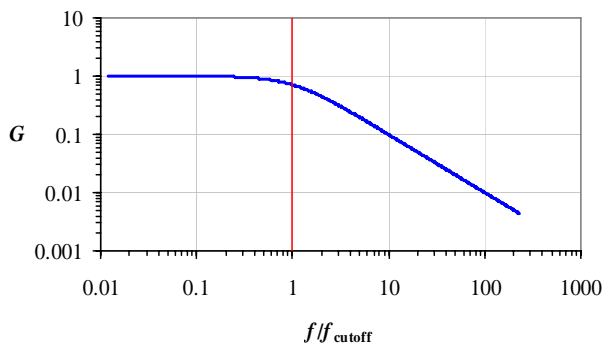
$$G = \frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_{\text{cutoff}}}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{f}{f_{\text{cutoff}}}\right)^2}}$$

- For this simple filter, G lies between 0 and 1.
- Note that the gain G of the filter can also be thought of as the ratio of the peak amplitude of V_{out} to the peak amplitude of V_{in} , or, multiplying numerator and denominator by two, G can be thought of as the ratio of the peak-to-peak amplitude of the output voltage to the peak-to-peak amplitude of the input voltage.
- The **phase shift** ϕ of the output signal is calculated as the angle of V_{out} in the complex plane,

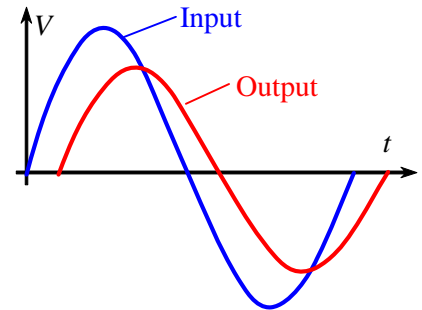
$$\phi = \arctan\left(\frac{\text{imaginary component}}{\text{real component}}\right) = \arctan\left(\frac{-\frac{\omega RC}{1 + (\omega RC)^2} V_{\text{in}}}{\frac{1}{1 + (\omega RC)^2} V_{\text{in}}}\right) = \arctan(-\omega RC) = \arctan\left(-\frac{\omega}{\omega_{\text{cutoff}}}\right). \text{ But}$$

since $\arctan(-x) = -\arctan(x)$, $\phi = -\arctan\left(\frac{\omega}{\omega_{\text{cutoff}}}\right)$. In terms of physical frequencies, $\phi = -\arctan\left(\frac{f}{f_{\text{cutoff}}}\right)$.

- Since V_{in} is a pure sine wave without any DC offset, recall that we write it as $V_{\text{in}} = V_p \sin(\omega t) = |V_{\text{in}}| \sin(\omega t)$, where $|V_{\text{in}}|$ is the peak amplitude of the input voltage signal ($|V_{\text{in}}| = V_p$).
- In similar fashion, V_{out} is also a pure sine wave without any DC offset, but *with a phase shift*; we write it as $V_{\text{out}} = |V_{\text{out}}| \sin(\omega t + \phi)$, where $|V_{\text{out}}|$ is the peak amplitude of the output voltage signal, as calculated above.
- What does all this mean *physically*, and why is this considered a *low-pass* filter circuit? Well, let us analyze what happens to both DC and AC voltage signals:
 - For a **DC signal**, $\omega = 0$, so $\omega/\omega_{\text{cutoff}} = 0$, and thus $|V_{\text{out}}| = |V_{\text{in}}|$, $G = 1$, and $\phi = 0$.
 - In other words, $V_{\text{out}} = V_{\text{in}}$.
 - *The low-pass filter circuit does not affect a DC signal at all.*
 - For a **low frequency AC signal**, $\omega \ll \omega_{\text{cutoff}}$, so $\omega/\omega_{\text{cutoff}} \ll 1$, and thus $|V_{\text{out}}| \approx |V_{\text{in}}|$, $G \approx 1$, and $\phi \approx 0$.
 - In other words, $V_{\text{out}} \approx V_{\text{in}}$, with **no significant phase shift**.
 - *Low frequency components pass through the low-pass filter circuit without much effect.*
 - This is why it is called a “low-pass” filter, by the way.
 - For an **AC signal with a frequency exactly equal to the cutoff frequency**, $\omega = \omega_{\text{cutoff}}$, so $\omega/\omega_{\text{cutoff}} = 1$, and thus $|V_{\text{out}}| = |V_{\text{in}}|/\sqrt{2}$, $G = 1/\sqrt{2} \cong 0.707$, and $\phi = -\arctan(1) = -\pi/4 = -45^\circ$.
 - In other words, $|V_{\text{out}}| = |V_{\text{in}}|/\sqrt{2}$, but with a **-45° phase shift**.
 - *A component of the input signal at exactly the cutoff frequency is attenuated by the circuit with a gain of $1/\sqrt{2} \cong 0.707$ (about 70.7%), and has a -45° phase shift.*
 - For a **high frequency AC signal**, $\omega \gg \omega_{\text{cutoff}}$, so $\omega/\omega_{\text{cutoff}} \gg 1$, and thus $|V_{\text{out}}| \rightarrow 0$, $G \rightarrow 0$, and, since $\arctan(\infty) = \pi/2$, $\phi \rightarrow -\pi/2 = -90^\circ$.
 - In other words, $V_{\text{out}} \rightarrow 0$, but with a **-90° phase shift**.
 - *High frequency components are attenuated or filtered by the low-pass filter circuit, and have a -90° phase shift.*
- Below are summary plots for this first-order low-pass filter, created in Excel, and plotted with log-log axes. On the left is the frequency response diagram, and on the right is the phase shift diagram. These plots are collectively called **Bode plots**.



- Notice how the low frequencies pass relatively unaffected, but the high frequencies get attenuated.
- Also notice that the drop-off with frequency looks linear on this log-log plot.
- The phase angle is zero (no phase shift) for very low frequencies, but falls off rapidly – at the cutoff frequency, the phase shift is already significant (-45°).
- For high frequencies the phase shift asymptotes to -90° .
- The phase shift is illustrated in another way on plot shown to the right, where the input and output signals are plotted for comparison, at some arbitrary frequency of the sine wave.
- Notice three things:
 - The frequency of the output signal is the same as that of the input signal.
 - The output signal is smaller in magnitude than the input signal, since the gain is less than 1.
 - The phase of the signal has shifted. Since the phase shift is *negative*, we say that **the output lags the input** (the peak occurs at a later time).



• **Example:**

Given: Noise at 1000 Hz is superimposed on a “carrier” frequency of 10 Hz. It is desired to apply a first-order passive low-pass filter to remove the noise so that only the carrier signal remains.

To do: (a) Choose the cutoff frequency of the low-pass filter. (b) If a capacitor with capacitance of $0.10 \mu\text{F}$ is available, what resistor should be used? (c) How much of the noise is reduced by this filter?

Solution:

(a) Obviously, the cutoff frequency must lie somewhere between 10 and 1000 Hz. If we pick a cutoff frequency too close to 10 Hz, some of the desired signal is attenuated. On the other hand, if the cutoff frequency is too big, the attenuation of 1000 Hz noise may not be enough. Let’s pick 50 Hz as a reasonable choice for the cutoff frequency. Answer: $f_{\text{cutoff}} = 50 \text{ Hz}$.

(b) From the previous discussion, we know that $f_{\text{cutoff}} = \frac{\omega_{\text{cutoff}}}{2\pi} = \frac{1}{2\pi RC}$, which we solve for resistance R ,

$$\text{yielding } R = \frac{1}{2\pi f_{\text{cutoff}} C}. \text{ Substitution yields } R = \frac{1}{2\pi \left(50 \frac{1}{\text{s}}\right) (0.10 \times 10^{-6} \text{ F})} \left(\frac{1 \text{ F}}{1 \text{ C/V}}\right) \left(\frac{1 \text{ C/s}}{1 \text{ A}}\right) \left(\frac{1 \Omega}{1 \text{ V/A}}\right) =$$

$3.1831 \times 10^4 \Omega$. Answer to two significant digits: $R = 3.2 \times 10^4 \Omega = 32. \text{ k}\Omega$. (Notice the unity conversion factors in the calculation.)

(c) To answer this question, we enter the noise frequency (1000 Hz) and the cutoff frequency (50 Hz) into

$$\text{the equation for the gain } G \text{ of the filter, } G = \frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{1}{\sqrt{1 + \left(\frac{f}{f_{\text{cutoff}}}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{1000}{50}\right)^2}} = 0.049938. \text{ Answer}$$

to two significant digits: $G = 0.050$. In other words, the noise is reduced by a factor of about 20.

Discussion: Although we have significantly reduced the noise, we cannot totally eliminate it; a higher-order low-pass filter would reduce the noise even more effectively.

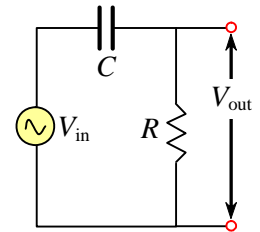
- It is standard to express the gain of both amplifiers and filters in terms of *decibels*,

$$G_{dB} = 20 \log_{10} G = 20 \log_{10} \frac{|V_{out}|}{|V_{in}|}. \text{ In the above example, } G_{dB} = 20 \log_{10}(0.049938) = -26. \text{ dB.}$$

- It is useful to also calculate the gain imposed by this filter on the 10 Hz signal. (Recall that the 10 Hz signal is the *desired* signal here.) Using the same equations as above, but with $f = 10$ Hz, we calculate the gain of the low-pass filter at a frequency of 10 Hz to be $G = 0.9806$, or $G_{dB} = 20 \log_{10}(0.9806) = -0.17$ dB.
- Is this filter adequate? The answer depends on the application. With this filter, the noise at 1000 Hz is reduced by a factor of 20, while the desired signal itself (10 Hz) is reduced by about 2 percent. For most applications, this is fine. However, if more attenuation of the noise is required, or if a two percent reduction of the signal is too much, then this filter would not be adequate. In such a case, the engineer would choose a higher-order filter, as discussed later.

First-order passive high-pass filter

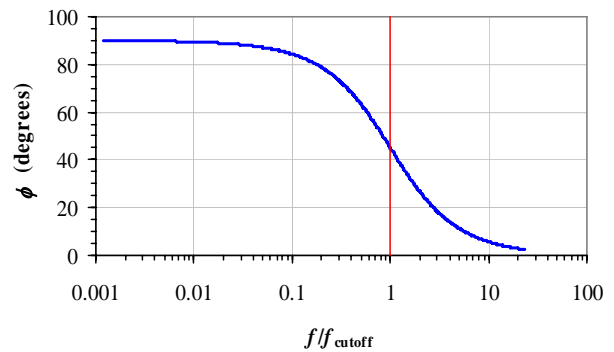
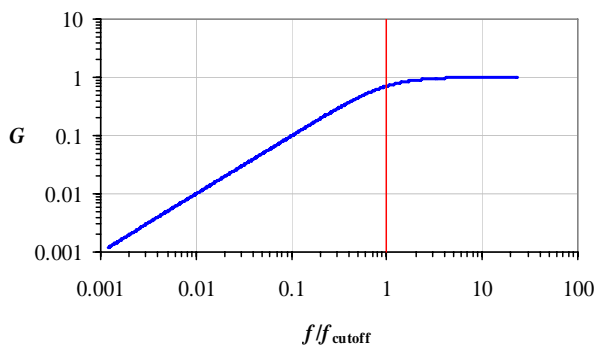
- A resistor and capacitor can be used together to create a simple *first-order passive high-pass filter circuit*. For a high-pass filter, the resistor and capacitor of the low-pass filter switch positions as shown to the right.
- The algebra to determine the filter function and the phase shift is similar to that performed above for the low-pass filter, and is not shown in detail here.
- Below is a summary of the equations for the output voltage signal for this simple passive first-order high-pass filter, noting that $V_{in} = |V_{in}| \sin(\omega t)$ and $V_{out} = |V_{out}| \sin(\omega t + \phi)$, as previously:



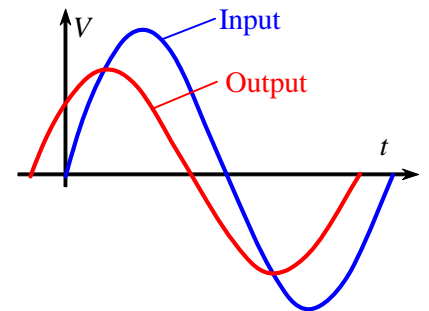
$$G = \frac{|V_{out}|}{|V_{in}|} = \frac{1}{\sqrt{1 + \left(\frac{\omega_{cutoff}}{\omega}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{f_{cutoff}}{f}\right)^2}} \text{ and } \phi = \arctan\left(\frac{\omega_{cutoff}}{\omega}\right) = \arctan\left(\frac{f_{cutoff}}{f}\right), \text{ where the cutoff}$$

frequency itself is the same as defined previously for the low-pass filter, namely, $f_{cutoff} = \frac{\omega_{cutoff}}{2\pi} = \frac{1}{2\pi RC}$.

- We again analyze what happens to both DC and AC voltage signals, to help us understand what a high-pass filter is doing to the signal:
 - For a **DC signal**, $\omega = 0$, $\omega_{cutoff}/\omega \rightarrow \infty$, $|V_{out}| \rightarrow 0$, $G \rightarrow 0$, and $\phi \approx \pi/2 = 90^\circ$.
 - In other words, **$V_{out} = 0$** . (It has a 90° phase shift, but this is inconsequential since the output is zero.)
 - The high-pass filter circuit completely cuts off or removes a DC signal.**
 - For a **low frequency AC signal**, $\omega \ll \omega_{cutoff}$, $\omega_{cutoff}/\omega \gg 1$, $|V_{out}| \rightarrow 0$, $G \rightarrow 0$, and $\phi \rightarrow \pi/2 = 90^\circ$.
 - In other words, **$V_{out} \approx 0$** , but with a **90° phase shift**.
 - Low frequency components are attenuated or filtered by the high-pass filter circuit, and have a 90° phase shift.**
 - For an **AC signal with a frequency exactly equal to the cutoff frequency**, $\omega = \omega_{cutoff}$, $\omega_{cutoff}/\omega = 1$, $|V_{out}| = |V_{in}|/\sqrt{2}$, $G = 1/\sqrt{2} \approx 0.707$, and $\phi = \arctan(1) = \pi/4 = 45^\circ$.
 - In other words, **$|V_{out}| = |V_{in}|/\sqrt{2}$** , but with a **$45^\circ$ phase shift**.
 - A component of the input signal at exactly the cutoff frequency is attenuated by the circuit with a gain of $1/\sqrt{2} \approx 0.707$ (about 70.7%), and has a 45° phase shift.**
 - For a **high frequency AC signal**, $\omega \gg \omega_{cutoff}$, $\omega_{cutoff}/\omega \ll 1$, $|V_{out}| \approx |V_{in}|$, $G \approx 1$, and $\phi \approx 0$.
 - In other words, **$V_{out} \approx V_{in}$** , and there is **no significant phase shift**.
 - High frequency components pass through the high-pass filter circuit without much effect.**
 - This is why it is called a “high-pass” filter, by the way.
- Below are the Bode plots for this first-order high-pass filter, created in Excel, and plotted with log-log axes. On the left is the frequency response diagram, and on the right is the phase shift diagram.
 - The plots are basically opposite to those of the low-pass filter – the high frequencies pass through relatively unaffected, but the low frequencies get attenuated.
 - Also notice that the drop-off with frequency looks linear on this log-log plot.
 - The phase angle is zero (no phase shift) for very high frequencies, but rises rapidly at the frequency is reduced – at the cutoff frequency, the phase shift is already significant (45°).
 - For low frequencies the phase shift asymptotes to 90° .

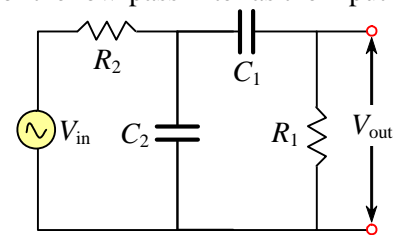


- The phase shift is illustrated in another way on the plot shown to the right, where the input and output signals are plotted for comparison, at some arbitrary frequency of the sine wave.
- Notice three things:
 - The frequency of the output signal is the same as that of the input signal.
 - The output signal is smaller in magnitude than the input signal, since the gain is less than 1.
 - The phase of the signal has shifted. Since the phase shift is *positive*, we say that *the output leads the input* (the peak appears to occur at an “earlier” time).



First-order passive band-pass filter

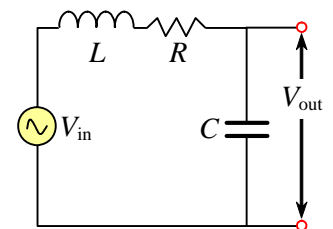
- A **first-order passive band-pass filter** circuit is created by using the output of the low-pass filter as the input for the high-pass filter (**low-pass filter and high-pass filter in series**), as shown to the right.
- The left portion of the schematic diagram represents the low-pass filter, and the right portion represents the high-pass filter.
- Here there are two cutoff frequencies to define – a low-pass cutoff frequency and a high-pass cutoff frequency:



- In terms of radian frequency, $\omega_{\text{cutoff}, 1} = \frac{1}{R_1 C_1}$ and $\omega_{\text{cutoff}, 2} = \frac{1}{R_2 C_2}$.
- In terms of physical frequency, $f_{\text{cutoff}, 1} = \frac{1}{2\pi R_1 C_1}$ and $f_{\text{cutoff}, 2} = \frac{1}{2\pi R_2 C_2}$.

Second-order passive low-pass filter

- If an inductor is added to the resistor and capacitor of a first-order low-pass filter, we create a higher-order low-pass filter, as shown in the circuit to the right. It turns out that this filter behaves as a **second-order passive low-pass filter**.
- Higher-order filters may introduce some “wiggles” in the time-response characteristics of the filter, due to second-order dynamic system behavior.



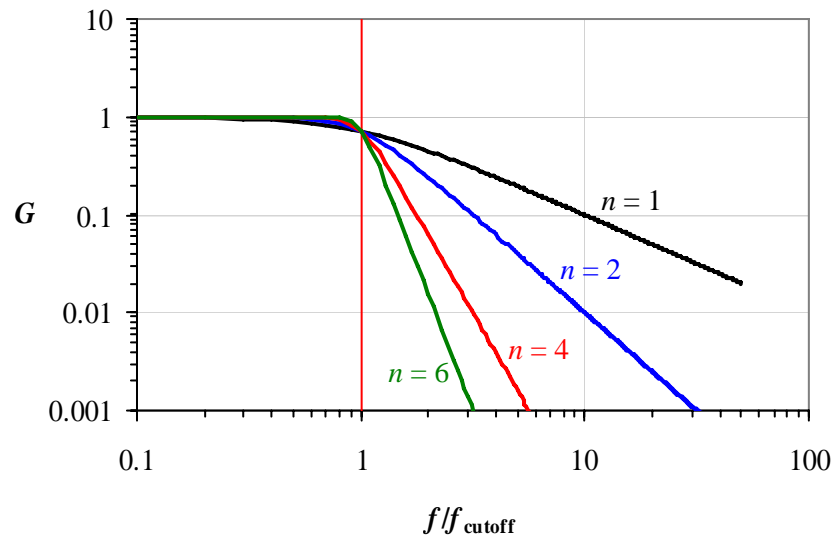
Higher-order low-pass filters

- Butterworth filters of higher order can be constructed, usually with op amps rather than simply with resistors, capacitors, and inductors – these are called **active filters**.
- In general, for a **Butterworth low-pass filter of order n**, the gain is

$$G = \frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_{\text{cutoff}}}\right)^{2n}}} = \frac{1}{\sqrt{1 + \left(\frac{f}{f_{\text{cutoff}}}\right)^{2n}}}$$

- Compared to a simple first-order low-pass filter, the equation for G is identical except that the exponent of the frequency ratio is $2n$ instead of 2. Thus, the simple passive first-order low-pass filter discussed earlier is a Butterworth low-pass filter of order 1 ($n = 1$).

- Higher-order filters have a much faster roll-off rate. For a given cutoff frequency, this means that the filter attenuates high frequencies much better, as illustrated in the plot below for Butterworth filters of order 1, 2, 4, and 6.



- For example, using the same numbers as in the example above (noise at $f = 1000$ Hz and $f_{\text{cutoff}} = 50$ Hz), the gain of a 4th-order low-pass Butterworth filter is $G = \frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{1}{\sqrt{1 + \left(\frac{f}{f_{\text{cutoff}}}\right)^{2(4)}}} = \frac{1}{\sqrt{1 + \left(\frac{1000}{50}\right)^8}} = 6.25 \times 10^{-6}$.
- Or, in terms of decibels, the gain is $G_{\text{dB}} = -104$ dB. Notice how much better is this attenuation, compared to that of a first-order filter, for which the gain was $G = 0.050$, or $G_{\text{dB}} = -26$ dB.
- Meanwhile, at the desired signal frequency of 10 Hz, the gain is 0.9999987 or -0.000011 dB, which is negligible.
- In other words, this filter greatly attenuates the 1000 Hz noise, but does not affect the 10 Hz component of the signal.
- Commercial Butterworth filters can be purchased with adjustable order, typically of order 1, 2, 4, 6, and 8.